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COMPARISON BETWEEN MLE AND BAYES ESTIMATOR OF SCALE PARAMETER OF GENERALIZED GAMMA DISTRIBUTION WITH KNOWN SHAPE PARAMETERS UNDER SQUARED ERROR LOSS FUNCTION

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Abstract

This papers presents the comparison between maximum likelihood estimator(MLE) and Bayes estimator of scale parameter of Generalized gamma distribution under Squared error loss function when shape parameters are known. Maximum likelihood estimator (MLE) of scale parameter is obtained. Using Jeffrey's prior , Bayes estimator of scale parameter is obtained under squared error loss function .For comparison purpose, a simulation study is also carried out to compute the relative efficiency of Bayes estimator with respect to maximum likelihood estimator.

Key Words: Jeffrey's Prior, Fisher Information, Squared Error Loss Function (SELF) , Maximum Likelihood Estimator (MLE), Posterior Expected Loss.

AMS Subject Classification: 62F10; 62F15; 65C05

1. Introduction

The generalized gamma distribution was first given by Stacy (1962). Most of the life time distributions such as Exponential, Gamma, Rayleigh , Pareto, Weibull [Bansal (2007),Sinha (1986),Sinha (1998) and Lawless (2003)] etc.used in reliability theory to represent life time behavior of the system are special cases of Generalized gamma distribution. Cohen (1969) considered it as a generalization of Weibull distribution. Harter (1967) used this distribution as a life time model. It is worth mentioning here that all these authors obtained estimators of parameters with complete samples. Recently, Krishna (2011) derived MLE and Bayes estimates of scale parameters of generalized gamma distribution with type II right censored scheme.
The probability density function (p.d.f.) of generalized gamma distribution with scale parameter θ and shape parameters (α, β) is given by

$$f(x,\theta,\alpha,\beta)=\frac{\beta}{\theta^{\alpha/\beta}\Gamma\left(\frac{\alpha}{\beta}\right)}x^{\alpha-1}e^{-x^{\beta}/\theta},x\geq 0;\theta,\alpha,\beta>0 \qquad (1)$$

It can be shown that exponential, gamma and weibull distributions are particular cases of (1) as :

(i) if we put (α, β) =1 in equation (1), we get exponential distribution with parameter θ,

(ii) for β =1,in equation(1),we get Gamma distribution with parameters (α, θ) and

(iii) for $\alpha = \beta = c$ and $\theta = \eta^c$ in equation(1),we get Weibull distribution with parameters (α, η).

In classical estimation approach, most widely used method of estimation is the method of maximum likelihood estimation (MLE).Bayesian method of estimation has also drawn attention in recent times. In Bayesian estimation, we combine the prior information and the sample information to get the posterior distribution and after that all decisions and conclusions based on posterior distribution are drawn. Another term that plays a vital role in Bayesian analysis is the loss function. There are several types of loss functions. But most frequently used loss function is the squared error loss function (SELF) and is defined by

$$L\left(\hat{\theta}, \theta\right) = (\hat{\theta} - \theta)^2 \tag{2}$$

Where $\hat{\theta}$ is an estimate of θ.This loss function is symmetric in nature i.e. it gives equal weight age to both over estimation and under estimation..Ahmed et al.(2010) considered ML estimation and Bayes estimation of scale parameter of Weibull distribution with known shape parameter and compared their performance under Squared error loss function. Keeping this in view, an attempt is made to compare the MLE and Bayes estimate of scale parameter under Squared error loss function.

The rest of the paper is structured as follows: In Section 2,under classical estimation, MLE of scale parameter θ is obtained. Section 3 deals with the Bayes estimation of θ under Squared error loss function. Section 4 presents the relative efficiency of Bayes estimator with respect to MLE. In Section 5, Monte Carlo simulation study is carried out to illustrate the results. Finally, we conclude the paper in Section 6.

2. Classical Estimation

In our study, we are interested with the Maximum Likelihood estimation procedure as one of the most important classical procedures.

2.1 Maximum Likelihood Estimation

Let $x_1.x_2 \ldots x$ be a random sample of size n drawn from Generalized gamma distribution with p.d.f. given by equation (1). Then likelihood function of the sample observations is given by

$$L = L\left(x \middle| \theta\right) = \frac{\beta^n}{\theta^{n\alpha/\beta}} \frac{1}{\left[\Gamma\left(\alpha/\beta\right)\right]^n} \left(\prod_{i=1}^{n} x_i^{\alpha-1}\right) e^{-\sum_{i=1}^{n} x_i^{\beta}/\theta} \tag{3}$$

Taking logarithm of both sides of equation (3), we get

$$\log L = n\log\beta - \frac{n\alpha}{\beta}\log\theta - n\log\Gamma\left(\alpha/\beta\right) + \left(\alpha-1\right)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n} x_i^{\beta}/\theta$$

The likelihood equation

$$\frac{\partial}{\partial\theta}\log L = 0$$

gives $\theta = \frac{\beta}{n\alpha}\sum_{i=1}^{n} x_i^{\beta}$

Thus MLE of θ is given by

$$\hat{\theta}_{ML} = \frac{\beta}{n\alpha}\sum_{i=1}^{n} x_i^{\beta} = \frac{\beta t}{n\alpha}, where, t = \sum_{i=1}^{n} x_i^{\beta}$$

And

$$\frac{\partial^2}{\partial\theta^2}\log L = \frac{n\alpha}{\beta\theta^2} - 2\sum_{i=1}^{n} x_i^{\beta}/\theta^3$$

$$I(\theta) = E\left(-\frac{\partial^2}{\partial\theta^2}\log L\right) = -\frac{n\alpha}{\beta\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n}\frac{\theta^{\alpha}}{\beta}$$

$$= -\frac{n\alpha}{\beta\theta^2} + \frac{2n\alpha}{\beta\theta^2}$$

$$= \frac{n\alpha}{\beta\theta^2} \qquad (4)$$

3. Bayesian Estimation

From Bayesian perspective choice of prior distribution of parameter and Loss function play an important role in estimation and prediction problems.. Based on knowledge and beliefs, researchers choose informative prior .However, an appropriate choice of prior is still a problem. Jeffrey (1946) proposed a rule according to which prior is given by

Prior $\propto \sqrt{I\left(\theta\right)}$

Where, I(θ) is the Fisher information on θ given in equation (4).Thus, prior distribution of θ is given by

$$g(\theta) \propto \sqrt{\frac{n\alpha}{\beta\theta^2}}$$

$$= k/\theta \text{ ,where k is a constant} \tag{5}$$

Now, combining the prior (5) with the likelihood function (3), the posterior distribution of θ is given by

$$\Pi(\theta|x) = \frac{L(x|\theta)g(\theta)}{\int_0^\infty L(x|\theta)g(\theta)d\theta} \tag{6}$$

The denominator of (6) is

$$\int_0^\infty L(x|\theta)g(\theta)d\theta = \frac{k\beta^n}{\left[\Gamma(\alpha/\beta)\right]^n}\left(\prod_{i=1}^n x_i^{\alpha-1}\right)\frac{\Gamma(n\alpha/\beta)}{\left(\sum_{i=1}^n x_i^\beta\right)^{n\alpha/\beta}}$$

Therefore, posterior distribution of θ is given by

$$\Pi(\theta|x) = \frac{\left(\sum_{i=1}^n x_i^\beta\right)^{n\alpha/\beta}}{\Gamma(n\alpha/\beta)\theta^{(n\alpha/\beta)+1}} e^{-\sum_{i=1}^n x_i^\beta/\theta}$$

$$= \frac{t^{\frac{n\alpha}{\beta}}}{\theta^{(n\alpha/\beta)+1}\Gamma(n\alpha/\beta)} e^{-t/\theta} \tag{7}$$

Which is the pdf of inverted- gamma distribution with parameters (nα/β ,t)

Where,

$t = \sum_{i=1}^n x_i^\beta \sim Gamma(\frac{n\alpha}{\beta}, \theta),$ since, X_i^β ,i=1,2,…n ,are independently and identically distributed gamma variates with parameters ($\frac{\alpha}{\beta}$,θ).

The p.d.f. of T= t is given by

$$h(t) = \frac{1}{\theta^{n\alpha/\beta}\Gamma\left(\frac{n\alpha}{\beta}\right)} t^{\frac{n\alpha}{\beta}-1} e^{-\frac{t}{\theta}}, t > 0 \tag{8}$$

4. The Relative efficiency of Bayes estimator($\hat{\theta}_{SB}$) with respect to MLE ($\hat{\theta}_{ML}$) under Squared error loss function (SELF)

In decision theoretic approach, the decision taken by a decision maker is a function which specifies $\hat{\theta}$ = d(x) as the action to be taken when the observed data is X=x. On taking action $\hat{\theta}$ =d(x), the decision maker incurs a loss $L(\hat{\theta},\theta)$ .A good decision is one that has minimum risk function(or minimum expected loss) given by

$$R(\hat{\theta},\theta) = E[L(\hat{\theta},\theta)]$$

$$= \int L(\hat{\theta},\theta) f(x,\theta)dx,$$ if X is a continuous random variable

The risk function of MLE under Squared error loss function is given by

$$R_{SELF}\left(\hat{\theta}_{ML},\theta\right) = E\left[\left(\hat{\theta}_{ML}-\theta\right)^2\right]$$

$$= E\left[\hat{\theta}_{ML}^2 + \theta^2 - 2\hat{\theta}_{ML}\theta\right]$$

$$= \int\left[\hat{\theta}_{ML}^2 + \theta^2 - 2\hat{\theta}_{ML}\theta\right] h(t)\, dt$$

$$= \int\left[\hat{\theta}_{ML}^2 + \theta^2 - 2\hat{\theta}_{ML}\theta\right] \frac{1}{\theta^{n\alpha/\beta}\Gamma\left(\frac{n\alpha}{\beta}\right)} t^{\frac{n\alpha}{\beta}-1} e^{-\frac{t}{\theta}} dt$$

$$= \frac{\theta^2\beta}{n\alpha} \tag{9}$$

We know that under Squared error loss function, Bayes estimator is the mean of posterior distribution (7) and is given by

$$\hat{\theta}_{SB} = E\left(\theta|x\right) = \frac{t}{\frac{n\alpha}{\beta}-1}$$

Now, the risk function of Bayes estimator under Squared error loss function is given by

$$R_{SELF}\left(\hat{\theta}_{SB},\theta\right)=E\left[\left(\hat{\theta}_{SB}-\theta\right)^2\right]$$

$$=E\left[\hat{\theta}_{SB}^2+\theta^2-2\hat{\theta}_{SB}\theta\right]$$

$$=\int\left[\hat{\theta}_{SB}^2+\theta^2-2\hat{\theta}_{SB}\theta\right]h(t)\,dt$$

$$=\int\left[\hat{\theta}_{SB}^2+\theta^2-2\hat{\theta}_{SB}\theta\right]\frac{1}{\theta^{n\alpha/\beta}\Gamma\left(\frac{n\alpha}{\beta}\right)}t^{\frac{n\alpha}{\beta}-1}e^{-\frac{t}{\theta}}dt$$

$$=\frac{\theta^2\beta\left(n\alpha+\beta\right)}{\left(n\alpha-\beta\right)^2} \tag{10}$$

As we know, the relative efficiency (R.E.) of estimator $\hat{\theta}_1$ with respect to (w.r.t.) estimator $\hat{\theta}_2$ is defined by

$$\text{R.E.}==\frac{R(\hat{\theta}_2,\theta)}{R(\hat{\theta}_1,\theta)}$$

Thus, if R.E.>1, then estimator $\hat{\theta}_1$ is better than estimator $\hat{\theta}_2$.

Therefore, the relative efficiency of Bayes estimator ($\hat{\theta}_{SB}$) w.r.t. MLE ($\hat{\theta}_{ML}$) under Squared error loss function is given by

$$\text{R.E}=\frac{R_{SELF}\left(\hat{\theta}_{ML},\theta\right)}{R_{SELF}\left(\hat{\theta}_{SB},\theta\right)} \tag{11}$$

5. Simulation Study

To compare the performance of MLE and Bayes estimators under Squared error loss function, we carry out simulation study. For this, we have generated N=1000 samples of sizes n=10,20,50,100 representing small, moderate and large samples.

Keeping the scale parameter θ=.5, 1.5, 2.5 and fixing the shape parameters α=..4 ,β=.8 and α=1.4,β=1.8 respectively the relative efficiency (11) of Bayes estimator with respect to MLE is computed .The values are shown in Table 1. All the calculations are performed on the package R 2.15.1.

6. Conclusion

According to the results obtained in Section 5, we observe from Table 1 that relative efficiency of Bayes estimator w.r.t. MLE under Squared error loss function is smaller than one for all sample sizes. However, it increases with the increase in sample size .Thus ,finally we conclude that ML estimator performs better than Bayes estimator. Therefore**,** in this scenario, the use of Maximum likelihood estimator is recommended.

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n	θ	α	β	$\hat{\theta}_{ML}$	$\hat{\theta}_{SB}$	$R_{SELF}\left(\hat{\theta}_{ML},\theta\right)$	$R_{SELF}\left(\hat{\theta}_{SB},\theta\right)$	*Relative efficiency*
10	0.5	.4	.8	0.4715008	0.4963166	0.01121698	0.01446009	0.7757202
		1.4	1.8	0.5032448	0.545687	0.02246551	0.03347366	0.67114
	1.5	.4	.8	0.726732	0.7649811	0.0266253	0.03432332	0.7757202
		1.4	1.8	2.316921	2.512324	0.4808245	0.7164295	0.67114
	2.5	.4	.8	0.8972021	0.9444233	0.04058129	0.05231434	0.7757202
		1.4	1.8	4.714145	5.111723	1.96523	2.928196	0.67114
20	.5	.4	.8	0.471278	0.483362	0.00557609	0.00632463	0.8816467
		1.4	1.8	0.5056683	0.5261288	0.01074163	0.01307811	0.8213444
	1.5	.4	.8	0.7302302	0.7489541	0.01338914	0.01518651	0.8816467
		1.4	1.8	2.338558	2.433182	0.2297229	0.2796913	0.8213444
	2.5	.4	.8	0.8929407	0.9158366	0.02001495	0.02270178	0.8816467
		1.4	1.8	4.771247	4.96430	0.9544918	1.162109	0.8213444
50	.5	.4	.8	0.4701855	0.4749349	0.00221452	0.00232841	0.9510852
		1.4	1.8	0.4990571	0.5069429	0.00398628	0.004310282	0.9248284
	1.5	.4	.8	0.7294848	0.7368533	0.00533117	0.005605348	0.9510852
		1.4	1.8	2.338535	2.375488	0.08771146	0.09484079	0.9248284
	2.5	.4	.8	0.8955204	0.904566	0.00803282	0.00844596	0.9510852
		1.4	1.8	4.7821	4.857664	0.367156	0.396999	0.9248284
100	.5	.4	.8	0.4713994	0.4737683	0.00111206	0.001140256	0.9752731
		1.4	1.8	0.5018481	0.505782	0.001990193	0.002069304	0.9617695
	1.5	.4	.8	0.7308634	0.7345361	0.002672982	0.002740752	0.9752731
		1.4	1.8	2.332116	2.350397	0.04298184	0.04469037	0.9617695
	2.5	.4	.8	0.8958892	0.9003911	0.004016478	0.004118311	0.9752731
		1.4	1.8	4.781447	4.818928	0.1808324	0.1880205	0.9617695

Table 1: Relative efficiency of Bayes estimator w.r.t. MLE under SELF